On algebraic integrability of the deformed elliptic Calogero–Moser problem

L. A. Khodarinova† and I. A. Prikhodsky‡

†*Magnetic Resonance Centre, School of Physics and Astronomy, University of Nottingham, Nottingham, England NG7 2RD, e-mail: LarisaKhodarinova@hotmail.com*

‡*Institute of Mechanical Engineering, Russian Academy of Sciences,M. Haritonievsky, 4, Centre, Moscow 101830 Russia*

Abstract.

Algebraic integrability of the elliptic Calogero–Moser quantum problem related to the deformed root systems $A_2(2)$ is proved. Explicit formulae for integrals are found.

Following to [1] (see also [2] and [3]) we call a Schrödinger operator

$$L = -\Delta + u(\vec{x}), \qquad \vec{x} \in R^n,$$

integrable if there exist n commuting differential operators $L_1 = L,\ L_2, \ldots, L_n$ with constant algebraically independent highest symbols $P_1\left(\vec{\xi}\right) = \left(\vec{\xi}\right)^2,\ P_2\left(\vec{\xi}\right), \ldots, P_n\left(\vec{\xi}\right)$, and *algebraically integrable* if there exists at least one more differential operator L_{n+1}, which commutes with the operators L_i, $i = 1, \ldots, n$, and whose highest symbol $P_{n+1}\left(\vec{\xi}\right)$ is also independent on x and takes different values on the solutions of the algebraic system

$$P_i\left(\vec{\xi}\right) = c_i, i = 1, \ldots, n, \tag{1}$$

for generic c_i.

The question how large is the class of the algebraically integrable Schrödinger operators is currently far from being understood, so any new example of such an operator is of substantial interest.

The main result of this paper is the proof of the algebraic integrability of the following Schrödinger operator

$$L=-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-m\frac{\partial^2}{\partial x_3^2}+2(m+1)\left(m\wp(x_1-x_2)+\wp(x_1-x_3)+\wp(x_2-x_3)\right) \tag{2}$$

when the parameter $m=2$. Here $\wp$ is the classical Weierstrass elliptic function satisfying the equation $\wp'^2-4\wp^3+g_2\wp+g_3=0$. The operator (2) was introduced by Chalykh, Feigin and Veselov in [4] and is related to the deformed root system $\mathbf{A_2(m)}$ (see [4] for details).

When $m=1$ this is the well-known three-particle Calogero–Moser problem. The usual integrability of this problem has been established by Calogero, Marchioro and Ragnisco in [5], the algebraic integrability has been proved in [6] and in a more general case in [7].

The deformed Calogero–Moser system (2) in the trigonometric and rational limits has been completely investigated by Chalykh, Feigin and Veselov in [4], where the algebraic integrability for the corresponding systems has been proved for any m. They have also conjectured that the same is true in the elliptic case.

In this paper we prove this conjecture for $m=2$. The first results in this direction have been found in [6], where it was proved that problem (2) is integrable. The corresponding integrals have the form

$$\begin{aligned}
&L_1=L=-\partial_1^2-\partial_2^2-m\partial_3^2+2(m+1)\left(m\wp_{12}+\wp_{13}+\wp_{23}\right),\\
&L_2=\partial_1+\partial_2+\partial_3,\\
&L_3=\partial_1\partial_2\partial_3+\left(\tfrac{1-m}{2}\right)(\partial_1+\partial_2)\partial_3^2+\left(\tfrac{1-m}{2}\right)\left(\tfrac{1-2m}{3}\right)\partial_3^3+\\
&\qquad+(m+1)\left(\wp_{23}\partial_1+\wp_{13}\partial_2\right)+m(m+1)\wp_{12}\partial_3+\\
&\qquad+\left(\tfrac{1-m}{2}\right)(m+1)\left((\wp_{13}+\wp_{23})\partial_3+\partial_3(\wp_{13}+\wp_{23})\right),
\end{aligned} \tag{3}$$

where we have used the notations $\partial_i=\partial/\partial x_i$, $\wp_{ij}=\wp(x_i-x_j)$.

It was also shown that the operator

$$\begin{aligned}
L_{12} \;=\;& (\partial_1-m\partial_3)^2(\partial_2-m\partial_3)^2\\
&-2(m+1)^2\wp_{23}(\partial_1-m\partial_3)^2-2(m+1)^2\wp_{13}(\partial_2-m\partial_3)^2\\
&+2m(m+1)\left(\wp_{12}-\wp_{13}-\wp_{23}\right)(\partial_1-m\partial_3)(\partial_2-m\partial_3)
\end{aligned} \tag{4}$$

$$
\begin{aligned}
&-m(m+1)\left(\wp_{12}'+m\wp_{13}'+3(m+1)\wp_{23}'\right)(\partial_1-m\partial_3)\\
&-m(m+1)\left(-\wp_{12}'+m\wp_{23}'+3(m+1)\wp_{13}'\right)(\partial_2-m\partial_3)\\
&-m(m+1)\wp_{12}''-3/2m^2(m+1)^2\wp_{13}''-3/2m^2(m+1)^2\wp_{23}''\\
&+m^2(m+1)^2\left(\wp_{12}^2+\wp_{13}^2+\wp_{23}^2\right)+2m(m+1)^2\left(\wp_{12}\wp_{13}+\wp_{12}\wp_{23}\right)\\
&+2(m+1)^2(2m^2+3m+2)\wp_{13}\wp_{23}
\end{aligned}
$$

commutes with L_1, L_2, L_3 and therefore is an additional integral of the problem (2). Unfortunately, this is not enough for algebraic integrability of the problem (2) since the highest symbol of L_{12} is invariant under permutation of ξ_1 and ξ_2 and therefore takes the same values on some of the solutions of the corresponding system (1).

In this paper we present an explicit formula of one more integral for the system (2) related to the deformed root systems $\mathbf{A_2(2)}$. This integral together with the previous integrals guarantees the algebraic integrability of the system (2) in case of $m=2$.

Theorem. *The system (2) with $m=2$ besides the quantum integrals given by (3) and (4) has also the following integral $L_{13}=I+I^*$, where*

$$
\begin{aligned}
I \;=\;& \tfrac{1}{2}(\partial_1-\partial_2)^4(\partial_1-2\partial_3)^2 \qquad (5)\\
&-9\wp_{13}(\partial_1-\partial_2)^4-24\wp_{12}(\partial_1-\partial_2)^2(\partial_1-2\partial_3)^2\\
&-6(\wp_{12}+\wp_{13}-\wp_{23})(\partial_1-\partial_2)^3(\partial_1-2\partial_3)\\
&+\Big(414\wp_{12}\wp_{13}+18\wp_{12}\wp_{23}+18\wp_{13}\wp_{23}\\
&\qquad+72\wp_{12}^2+108\wp_{13}^2+36\wp_{23}^2-\tfrac{201}{2}g_2\Big)(\partial_1-\partial_2)^2\\
&+\Big(144\wp_{12}\wp_{13}-144\wp_{12}\wp_{23}\\
&\qquad+432\wp_{12}^2+18\wp_{13}^2-18\wp_{23}^2+33g_2\Big)(\partial_1-\partial_2)(\partial_1-2\partial_3)\\
&+(288\wp_{12}^2-69g_2)(\partial_1-2\partial_3)^2-369\wp_{12}'\wp_{13}'+288\wp_{12}'\wp_{23}'+18\wp_{13}'\wp_{23}'\\
&-5760\wp_{12}^3-648\wp_{13}^3-288\wp_{23}^3-\wp_{12}^2(3834\wp_{13}+1350\wp_{23})\\
&+\wp_{13}^2(594\wp_{12}-594\wp_{23})-\wp_{23}^2(648\wp_{12}-324\wp_{13})\\
&+g_2\left(\tfrac{5085}{2}\wp_{12}+\tfrac{2061}{2}\wp_{13}+990\wp_{23}\right)
\end{aligned}
$$

and I^* is the operator adjoined to I.

The integral $L_4 = L_{13} + \frac{1}{2}L_{23}$, where L_{23} is given by the same formula (5) after permutation of x_1 and x_2 and ∂_1 and ∂_2, is an additional integral which together with L_1, L_2, L_3 guarantees the algebraic integrability.

Let us first comment on how this new integral L_{13} has been found. The highest symbol has been borrowed from the trigonometric case [4]. The commutativity relation between this integral and the Hamiltonian L_1 imposes a very complicated overdetermined system of relations on the coefficients of the integral. We have resolved these relations combining the direct analysis with the use of a computer. The addition theorem and the differential equations for the elliptic $\wp-$function play the essential role in these calculations. The fact that this overdetermined system has a solution seems to be remarkable.

It is obvious from the explicit formula that L_{13} commutes with L_2. The commutativity of L_{13} and L_3 has been checked with the help of a computer. We have used a special program, which has been created for this purpose, and the same technical tricks as in our previous paper [8]. The commutativity of the operators L_1, L_2, L_3 has been proved in [6].

It is easy to check that the highest symbol of L_4 takes different values on the solutions of the corresponding system (1). This completes the proof of the algebraic integrability.

Remark. We should mention that according to Krichever's general result (see [1]) integrals L_1, L_2, L_3, L_4 satisfy certain algebraic relations (spectral relations). In [8] we have found explicitly these relations in the non-deformed case $m = 1$. In the deformed case it seems to be a much more involved problem.

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